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Note

Some notes on the (q, t)-Stirling numbers

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ABSTRACT

The orthogonality of the (q, t)-version of the Stirling numbers has recently been proved by Cai and Readdy using a bijective argument. In this paper, we introduce new recurrences for the (q, t)-Stirling numbers and provide a (q, t)-analogue for sums of powers. Specializations of these results are given in terms of Stirling numbers or q-Stirling numbers.

1. Introduction

The q-Stirling numbers of the first kind $s_q[n, k]$ and the second kind $S_q[n, k]$ are a natural extension of the classical Stirling numbers. Recall that the q-Stirling numbers are the coefficients in the expansions

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n, k]x^k \qquad \text{and} \qquad x^n = \sum_{k=0}^{n} S_q[n, k](x)_{k,q}, \tag{1}$$

where

$$(x)_{n,q} = \prod_{k=0}^{n-1}(x - [k]_q)$$

is the q-analogue of the nth falling factorial of x (with $(x)_{0,q} = 1$) and

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q} \tag{2}$$

is q-number. There is a long history of studying q-Stirling numbers [1,3–5,7,13,16,17,20].

Combinatorial interpretations of the q-Stirling numbers have recently been given by Cai and Readdy in [2] using the more compact q-$(1 + q)$-analogues. In addition, Cai and Readdy considered the following extension for (2)

$$[n]_{q,t} = \begin{cases} (q^{n-2} + q^{n-4} + \cdots + 1) \cdot t, & \text{for } n \text{ even},\\ q^{n-1} + (q^{n-3} + q^{n-5} + \cdots + 1) \cdot t, & \text{for } n \text{ odd}, \end{cases} \tag{3}$$

where $t = 1 + q$ and generalized (1) to (q, t)-polynomials. We remark the following identity.

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Lemma 1. *For $n \geqslant 2$,*

$$[n]_{q,t} = q^2[n-2]_{q,t} + t,$$

with $[1]_{q,t} = 1$.

The (q, t)-Stirling numbers of the first kind $s_{q,t}[n, k]$ and the second kind $S_{q,t}[n, k]$ are the coefficients in the expansions

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k]x^k \qquad \text{and} \qquad x^n = \sum_{k=0}^{n} S_{q,t}[n,k](x)_{k,q,t}, \tag{4}$$

where

$$(x)_{n,q,t} = \prod_{k=0}^{n-1}(x - [k]_{q,t})$$

is the (q, t)-analogue of the nth falling factorial of x, with $(x)_{0,q,t} = 1$. Because

$$[n]_q = [n]_{q,1+q}$$

it is clear that the q-Stirling numbers are specializations of the (q, t)-Stirling numbers.

According to Cai and Readdy [2, Corollary 9.2], the (q, t)-analogues of Stirling numbers of the first and second kinds satisfy the following recurrences

$$s_{q,t}[n,k] = s_{q,t}[n-1,k-1] - [n-1]_{q,t} \cdot s_{q,t}[n-1,k]$$

and

$$S_{q,t}[n,k] = S_{q,t}[n-1,k-1] + [k]_{q,t} \cdot S_{q,t}[n-1,k],$$

with boundary conditions

$$s_{q,t}[n,0] = S_{q,t}[n,0] = \delta_{n,0} \qquad \text{and} \qquad s_{q,t}[0,k] = S_{q,t}[0,k] = \delta_{0,k},$$

where $\delta_{i,j}$ is the usual Kronecker delta function. Moreover, Cai and Readdy [2, Theorem 9.4] give a bijective argument to show that the (q, t)-Stirling numbers of the first and second kinds are orthogonal:

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \sum_{k=m}^{n} S_{q,t}[n,k] \cdot s_{q,t}[k,m] = \delta_{m,n}.$$

In this paper, motivated by these results, we shall provide other properties of the (q, t)-Stirling numbers that are similar to those of the classical Stirling numbers or the q-Stirling numbers.

2. (q, t)-Stirling recurrences

The classical (signed) Stirling numbers of the first kind

$$s(n,k) = s_{1,2}[n,k]$$

and the Stirling number of the second kind

$$S(n,k) = S_{1,2}[n,k]$$

are specializations of the (q, t)-Stirling numbers. It is well known that the Stirling numbers satisfy the following recurrence relations [10, Eq. (2) p. 186]

$$s(n+1,k+1) = \sum_{j=k}^{n}(-1)^{j-k}\binom{j}{k}s(n,j)$$

and [10, Eq. (9) p. 187]

$$S(n+1,k+1) = \sum_{j=k}^{n}\binom{n}{j}S(j,k).$$

In this section, we shall prove a similar result for the (q, t)-version of the Stirling numbers.

Theorem 1. *The (q,t)-Stirling numbers of the first and second kinds satisfy the following recurrences:*

(i) $s_{q,t}[n+2,k+2]=\sum_{i=k}^{n}(-1)^{i-k}t^{i-k-1}q^{2n-2i}\left(q+\frac{i+1}{k+1}\right)\binom{i}{k}s_{q,t}[n,i]$,

(ii) $S_{q,t}[n+2,k+2]=S_{q,t}[n+1,k+2]+\sum_{i=k}^{n}t^{n-i}q^{2i-2k}\binom{n}{i}S_{q,t}[i,k]$.

Proof. Recall [12] that the kth elementary symmetric function $e_k(x_1,x_2,\dots,x_n)$ and the kth complete homogeneous symmetric function $h_k(x_1,x_2,\dots,x_n)$ are given, respectively, by

$$e_k(x_1,x_2,\dots,x_n)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}x_{i_1}x_{i_2}\dots x_{i_k},$$

$$h_k(x_1,x_2,\dots,x_n)=\sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k\leqslant n}x_{i_1}x_{i_2}\dots x_{i_k},$$

for $k=1,2,\dots,n$. We set $e_0(x_1,\dots,x_n)=1$ and $h_0(x_1,\dots,x_n)=1$ by convention. Taking into account (4) it is an easy exercise to show that the (q,t)-Stirling numbers are specializations of the elementary and complete homogeneous symmetric functions [12, Ch. I, S. 2, Example 11], i.e.,

$$s_{q,t}[n,n-k]=(-1)^k e_k\left([1]_{q,t},[2]_{q,t},\dots,[n-1]_{q,t}\right) \tag{5}$$

and

$$S_{q,t}[n+k,n]=h_k\left([1]_{q,t},[2]_{q,t},\dots,[n]_{q,t}\right). \tag{6}$$

To prove this theorem, we consider the relations

$$e_k(x_1+1,\dots,x_n+1)=\sum_{i=0}^{k}\binom{n-i}{k-i}e_i(x_1,\dots,x_n) \tag{7}$$

and

$$h_k(x_1+1,\dots,x_n+1)=\sum_{i=0}^{k}\binom{n-1+k}{k-i}h_i(x_1,\dots,x_n), \tag{8}$$

which are special cases of the following Schur function formula [12, p. 47, Example 10]

$$s_\lambda(x_1+1,\dots,x_n+1)=\sum_{\mu\subseteq\lambda}d_{\lambda\mu}s_\mu(x_1,\dots,x_n),$$

where

$$d_{\lambda\mu}=\det\left(\binom{\lambda_i+n-i}{\mu_j+n-j}\right)_{1\leqslant i,j\leqslant n}$$

is a determinant of a matrix of binomial coefficients built from the parts of the integer partitions λ and μ. By (7) and (8), with x_i replaced by x_i/t $(t\neq 0)$, we obtained

$$e_k(x_1+t,\dots,x_n+t)=\sum_{i=0}^{k}\binom{n-i}{k-i}e_i(x_1,\dots,x_n)t^{k-i} \tag{9}$$

and

$$h_k(x_1+t,\dots,x_n+t)=\sum_{i=0}^{k}\binom{n-1+k}{k-i}h_i(x_1,\dots,x_n)t^{k-i}. \tag{10}$$

Considering Lemma 1 and the relation (9), we can write

$$\begin{aligned}
&e_k\left([1]_{q,t},[2]_{q,t},\dots,[n+1]_{q,t}\right)\\
&\quad=e_k\left([2]_{q,t},[3]_{q,t},\dots,[n+1]_{q,t}\right)+e_{k-1}\left([2]_{q,t},[3]_{q,t},\dots,[n+1]_{q,t}\right)\\
&\quad=e_k\left(t,q^2[1]_{q,t}+t,\dots,q^2[n-1]_{q,t}+t\right)\\
&\qquad+e_{k-1}\left(t,q^2[1]_{q,t}+t,\dots,q^2[n-1]_{q,t}+t\right)\\
&\quad=\sum_{i=0}^{k}\binom{n-i}{k-i}e_i\left([1]_{q,t},\dots,[n-1]_{q,t}\right)q^{2i}t^{k-i}
\end{aligned}$$

$$+\sum_{i=0}^{k-1}\binom{n-i}{k-1-i}e_i\left([1]_{q,t},\ldots,[n-1]_{q,t}\right)q^{2i}t^{k-1-i}$$

$$=\sum_{i=0}^{k}\left[t\binom{n-i}{k-i}+\binom{n-i}{k-1-i}\right]e_i\left([1]_{q,t},\ldots,[n-1]_{q,t}\right)q^{2i}t^{k-1-i}.$$

In terms of the q-Stirling numbers of the first kind, this relation can be written as

$$s_{q,t}[n+2,n+2-k]=\sum_{i=0}^{k}(-1)^{k-i}\left[t\binom{n-i}{k-i}+\binom{n-i}{k-1-i}\right]s_{q,t}[n,n-i]q^{2i}t^{k-1-i}.$$

By this identity, with k replaced by $n-k$, we obtain

$$\begin{aligned}
&s_{q,t}[n+2,k+2]\\
&\quad=\sum_{i=0}^{n-k}(-1)^{n-k-i}\left[t\binom{n-i}{n-k-i}+\binom{n-i}{n-k-1-i}\right]s_{q,t}[n,n-i]q^{2i}t^{n-k-1-i}\\
&\quad=\sum_{i=k}^{n}(-1)^{n-i}\left[t\binom{n+k-i}{n-i}+\binom{n+k-i}{n-1-i}\right]s_{q,t}[n,n+k-i]q^{2i-2k}t^{n-1-i}\\
&\quad=\sum_{i=k}^{n}(-1)^{i-k}\left[t\binom{i}{k}+\binom{i}{k+1}\right]s_{q,t}[n,i]q^{2n-2i}t^{i-k-1}.
\end{aligned}$$

The proof of the first identity follows easily.

The proof of the second identity is quite similar to the proof of the first identity. So considering Lemma 1 and the relation (10), we can write

$$\begin{aligned}
&h_k\left([1]_{q,t},[2]_{q,t},\ldots,[n+2]_{q,t}\right)\\
&\quad=h_{k-1}\left([1]_{q,t},[2]_{q,t},\ldots,[n+2]_{q,t}\right)+h_k\left([2]_{q,t},[3]_{q,t},\ldots,[n+2]_{q,t}\right)\\
&\quad=h_{k-1}\left([1]_{q,t},[2]_{q,t},\ldots,[n+2]_{q,t}\right)+h_k\left(t,q^2[1]_{q,t}+t,\ldots,q^2[n]_{q,t}+t\right)\\
&\quad=h_{k-1}\left([1]_{q,t},[2]_{q,t},\ldots,[n+2]_{q,t}\right)+\sum_{i=0}^{k}\binom{n+k}{k-i}h_i\left([1]_{q,t},\ldots,[n]_{q,t}\right)q^{2i}t^{k-i}.
\end{aligned}$$

This relation can be rewritten in terms of the (q,t)-Stirling numbers of the second kind as follows:

$$S_{q,t}[n+2+k,n+2]=S_{q,t}[n+1+k,n+2]+\sum_{i=0}^{k}\binom{n+k}{k-i}S_{q,t}[n+i,n]q^{2i}t^{k-i}.$$

Replacing n with $n-k$ gives

$$S_{q,t}[n+2,n+2-k]=S_{q,t}[n+1,n+2-k]+\sum_{i=0}^{k}\binom{n}{k-i}S_{q,t}[n-k+i,n-k]q^{2i}t^{k-i}.$$

Replacing k with $n-k$ gives

$$\begin{aligned}
S_{q,t}[n+2,k+2]&=S_{q,t}[n+1,k+2]+\sum_{i=0}^{n-k}\binom{n}{n-k-i}S_{q,t}[k+i,k]q^{2i}t^{n-k-i}\\
&=S_{q,t}[n+1,k+2]+\sum_{i=k}^{n}\binom{n}{n-i}S_{q,t}[i,k]q^{2i-2k}t^{n-i}.
\end{aligned}$$

The second identity is proved. □

New recurrence relations can be easily derived for Stirling numbers or q-Stirling numbers considering Theorem 1.

Corollary 1. *The q-Stirling numbers of the first and second kinds satisfy the following recurrences:*

(i) $s_q[n+2,k+2]=\sum_{i=k}^{n}(-1-q)^{i-k-1}q^{2n-2i}\left(-q-\frac{i+1}{k+1}\right)\binom{i}{k}s_q[n,i]$,

(ii) $S_q[n+2,k+2]=S_q[n+1,k+2]+\sum_{i=k}^{n}(1+q)^{n-i}q^{2i-2k}\binom{n}{i}S_q[i,k]$.

Corollary 2. *The Stirling numbers of the first and second kinds satisfy the following recurrences:*

(i) $s(n+2,k+2) = \frac{1}{2(k+1)} \sum_{i=k}^{n} (-2)^{i-k}(k+i+2)\binom{i}{k} s(n,i)$,

(ii) $S(n+2,k+2) = S(n+1,k+2) + \sum_{i=k}^{n} 2^{n-i}\binom{n}{i} S(i,k)$.

3. A (q,t)-analogue for sums of powers

Faulhaber's formula [11], expresses the sum of the kth powers of the first n positive integers as a $(k+1)$th-degree polynomial function of n as follows:

$$\sum_{j=1}^{n} j^k = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} B_j(0) n^{k+1-j}, \tag{11}$$

where $B_n(x)$ are Bernoulli polynomials,

$$B_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(x+j)^n}{k+1}.$$

The problem of q-analogues of the sums of powers has been approached by several authors [6,18,21]. They have found q-analogues of the Faulhaber formula (11) for $k = 1, 2, \ldots, 5$. A general formula was provided in 2005 by Guo and Zeng [9]:

$$S_{2k+1,n}(q) = \sum_{j=0}^{k} (-1)^j P_{k,j}(q) \frac{(1-q^n)^{k+1-j}(1-q^{n+1})^{k+1-j} q^{jn}}{(1-q^2)(1-q)^{2k-3j} \prod_{i=0}^{j}(1-q^{k+1-i})}$$

and

$$S_{2k,n}(q) = \sum_{j=0}^{k} (-1)^j Q_{k,j}(q^{\frac{1}{2}}) \frac{(1-q^{n+\frac{1}{2}})(1-q^n)^{k-j}(1-q^{n+1})^{k-j}(1-q^{\frac{1}{2}})^j q^{jn}}{(1-q^2)(1-q)^{2k-2j-1} \prod_{i=0}^{j}(1-q^{k-i+\frac{1}{2}})},$$

where

$$S_{k,n}(q) = \sum_{j=1}^{n} \frac{1-q^{2j}}{1-q^2} \left(\frac{1-q^j}{1-q}\right)^{k-1} q^{(n-j)(k+1)/2}.$$

The polynomials $P_{k,j}(q)$ and $Q_{k,j}(q)$ are given explicitly in [9, Theorems 1.1 and 1.2]. Very recently, Merca [15] introduced a q-analogue for sums of powers in terms of the q-Stirling numbers of both kinds, i.e.,

$$\sum_{j=1}^{n} \left(\frac{1-q^j}{1-q}\right)^k = -\sum_{j=1}^{k} j \cdot s_q[n+1, n+1-j] \cdot S_q[n+k-j, n].$$

In this section, motivated by these results, we shall provide a (q,t)-analogue for the sums of powers in terms of the (q,t)-Stirling numbers of both kinds. For what follows, let η be the characteristic function of the set of odd numbers, i.e.,

$$\eta(n) = \begin{cases} 1, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases}$$

Theorem 2. *Let k and n be positive integers. Then*

$$\sum_{j=1}^{n} \left(\eta(j) \cdot q^{j-1} + t \cdot [\lfloor j/2 \rfloor]_{q^2}\right)^k = -\sum_{j=1}^{k} j \cdot s_{q,t}[n+1, n+1-j] \cdot S_{q,t}[n+k-j, n].$$

Proof. The proof is similar to the proof of Merca [15, Theorem 1.1]. □

4. Two identities involving (q,t)-Stirling numbers

Recall that the q-analogues of the binomial coefficients are the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{[n]_q!}{[k]_q![n-k]_q!}, & \text{for } k \in \{0, \ldots, n\}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

is the q-factorial, with $[0]_q! = 1$. Guo and Yang obtained in [8] a q-analogue of some binomial coefficient identities of Sun [19] as follows:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_q q^{\binom{n-2k}{2}} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q,$$

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_q q^{\binom{n-4k}{2}} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_q.$$

Merca provided in [14] generalizations of these identities and obtained new convolutions for the complete and elementary symmetric functions:

$$\sum_{\substack{i+j+k=n \\ i+j \text{ even}}} (-1)^i h_i(x_1,\ldots,x_n) h_j(x_1,\ldots,x_n) e_k(x_1,\ldots,x_n) = h_n(x_1,\ldots,x_n)$$

and

$$\sum_{\substack{i+j+k=n \\ i+j \text{ even}}} (-1)^i e_i(x_1,\ldots,x_n) e_j(x_1,\ldots,x_n) h_k(x_1,\ldots,x_n) = e_n(x_1,\ldots,x_n).$$

In this way, considering that the (q,t)-Stirling numbers are specializations of the elementary and complete homogeneous symmetric functions, we derive the following result.

Theorem 3. *The (q,t)-Stirling numbers of the first and second kinds satisfy the following relations:*

(i) $$\sum_{\substack{i+j+k=n \\ i+j \text{ even}}} (-1)^{i+k} S_{q,t}[n+i,n] S_{q,t}[n+j,n] s_{q,t}[n+1,n+1-k] = S_{q,t}[2n,n],$$

(ii) $$\sum_{\substack{i+j+k=n-1 \\ i+j \text{ even}}} (-1)^j s_{q,t}[n,n-i] s_{q,t}[n,n-j] S_{q,t}[n-1+k,n-1] = s_{q,t}[n,1].$$

5. Concluding remarks

The complete and elementary symmetric functions have been used in this paper to discover and prove new identities involving the (q,t)-Stirling numbers. Specializations of these results can be given in terms of the Stirling numbers or the q-Stirling numbers.

The first identity of Theorem 1 can be rewritten in terms of the unsigned (q,t)-Stirling numbers

$$c_{q,t}[n,k] = (-1)^{n-k} s_{q,t}[n,k]$$

as follows

$$c_{q,t}[n+2,k+2] = \sum_{i=k}^{n} t^{i-k-1} q^{2n-2i} \left(q + \frac{i+1}{k+1} \right) \binom{i}{k} c_{q,t}[n,i].$$

This expression would very likely lend itself to a bijective proof using the bistatistic on the restricted staircase-shaped boards of Cai and Readdy as it no longer has negatives appearing in it.

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